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On the Extreme Relativistic Limit of Arbitrary Spin Wave Equations

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Abstract

The extreme relativistic limit (E-representation) of the wave equation in the Schrödinger form $i\partial\psi/\partial t = H\psi$ describing particles and anti-particles of spin 's' and non-zero rest mass 'm' is presented here. As the wave function has just the minimum number of $2(2s+1)$ components, the necessity of avoiding redundant components by auxiliary conditions does not arise. Relevant expressions are given for the infinitesimal generators of the Poincaré group and for the operators representing the observables in this representation.

1. *Introduction*

In recent years Weaver *et al.* (1964) and Mathews (1966b) have studied relativistic wave equations for arbitrary spin of non-zero rest mass in the Schrödinger form,

$$i\frac{\partial\psi}{\partial t} = H\psi \tag{1.1}$$

where the locally covariant wave function ψ transforms according to the $D(0,s) \oplus D(s,0)$ representation of the Lorentz group. The advantage of this formulation is that the wave function has just the minimum number of components, $2(2s+1)$, required to describe particles–anti-particles of spin 's' and non-zero rest mass. The relativistic invariance of equation (1.1) is ensured by requiring that H and $i(\partial/\partial t)$ have identical commutation relations with generators of the Poincaré group defined over the space of wave function ψ.

$$[H, \mathbf{P}] = 0 \tag{1.2}$$

$$[H, \mathbf{J}] = 0; \qquad \mathbf{J} = (\mathbf{x} \times \mathbf{p}) + \mathbf{S}, \qquad \text{with} \qquad \mathbf{S} = \begin{pmatrix} \mathbf{s} & 0 \\ 0 & \mathbf{s} \end{pmatrix} \tag{1.2a}$$

$$[H, \mathbf{K}] = i\mathbf{p} \tag{1.2b}$$

$$\mathbf{K} = t\mathbf{p} - \mathbf{x}H + i\boldsymbol{\lambda}, \qquad \text{with} \qquad \boldsymbol{\lambda} = \begin{pmatrix} \mathbf{s} & 0 \\ 0 & -\mathbf{s} \end{pmatrix} = \rho_3 \mathbf{S} \tag{1.2c}$$

Here $\mathbf{s} = (s_1, s_2, s_3)$ is the $(2s+1)$ dimensional angular momentum matrix.

These conditions on 'H' and second quantisability of the wave equation (1.1) consistent with microcausality permit only the following two Hamiltonians (Mathews, 1967), the first being applicable for half integral spin and the second for integer spins.

$$H = \sum_{1/2}^{s} E \tanh 2\mu\theta C_\mu + \rho_1 \sum_{1/2}^{s} E \operatorname{sech} 2\mu\theta B_\mu \tag{1.3}$$

$$H = \sum_{0}^{s} E \coth 2\mu\theta C_\mu + \rho_1 \sum_{0}^{s} E \operatorname{cosech} 2\mu\theta C_\mu \tag{1.3a}$$

where

$$\cosh\theta = E/m; \qquad \sinh\theta = p/m \tag{1.3b}$$

ρ_1 and ρ_3 are Pauli's first and third matrices whose elements are taken as $(2s+1)$ dimensional matrices rather than just numbers. B_μ and C_μ are even and odd combinations of the projection operator Λ_μ to the eigenvalue μ of $\lambda p = (\boldsymbol{\lambda}.\mathbf{p})/p$.

$$B_\mu = \Lambda_\mu + \Lambda_{-\mu} \tag{1.4}$$

$$C_\mu = \Lambda_\mu - \Lambda_{-\mu} \tag{1.4a}$$

$$B_\mu B_\nu = C_\mu C_\nu = B_\mu \delta_{\mu\nu} \tag{1.4b}$$

$$B_\mu C_\nu = C_\mu \delta_{\mu\nu} \tag{1.4c}$$

However, for describing the motion of particles in the non-relativistic or the extreme relativistic limits, one makes use of certain special representations related to the above, i.e. (1.1), by similarity transformations. A generalisation of the Foldy & Wouthuysen (1950) transformation (given by them for spin-half) has been adopted by Mathews (1966a) and Sankaranarayanan & Good (1965) to get at the canonical representation in which the position and spin operators are represented by $\mathbf{x}$ and S. This canonical representation, being the non-relativistic limit of the wave equation (1.1), has been found especially suitable for obtaining the electromagnetic properties of a spin-one particle by Shay & Good (1969). We present in this paper another representation which would be most appropriate for the description of particles moving with extremely relativistic speeds. The discussion of such a representation in the spin-half case was presented by Cini & Touscheck (1958) years ago by projecting equation (1.1) to the extreme relativistic limit, and since then its generalisation has been attempted (Mathews & Sankaranarayanan, 1961, 1962, 1964) in special cases of higher spins. But the analysis presented here is very general as it is applicable for any spin and at the same time does not suffer from the necessity of eliminating redundant components as in the conventional manifestly covariant equations. Also, explicit expressions have been obtained for the generators of the Poincaré group. To our knowledge this is the first time that such a general derivation has been presented for arbitrary spin. For convenience, following Bose *et al.* (1961), we shall call equation (1.1), with

Poincaré generators given by (1.2) and (1.3), the D-representation. The Foldy–Wouthuysen representation will be termed 'C'-representation and the extreme relativistic one as the 'E'-representation. The procedure adopted here, to go to the 'E'-representation, is as follows: From the knowledge of the transformation operator linking the representations C and D, an operator is constructed to transform the C-representation into the E-representation and that operator is used to obtain relevant expressions for the infinitesimal generators of the Poincaré group and to construct a Lorentz invariant scalar product. Suitable well-behaved operators for the observables in the E-representation have also been obtained and finally it is shown that these operators can be expressed purely in terms of the generators of the Poincaré group. Since the form of H in D-representation is different for integer and half-integer spins we shall accordingly deal with them separately.

2. *The C- and E-Representations for Half-Integral Spins*

(i) *C-Representation*

The wave equation is of the form

$$i\partial\psi_C/\partial t = H_C\psi_C \tag{2.1}$$

$$\psi_C = S^{-1}\psi = \left[\sum_{1/2}^{s} \beta_\mu^{(+)} B_\mu + \rho_1 \sum_{1/2}^{s} \beta_\mu^{(-)} C_\mu\right]\psi \tag{2.1a}$$

$$\beta_\mu^{(\pm)} = \frac{\sqrt{(E)}\,(mE+mp)^\mu\,[(E+p)^{2\mu} \pm m^{2\mu}]}{\sqrt{(m)}\,[(E+p)^{4\mu}+m^{4\mu}]} \tag{2.1b}$$

$$H_C = \rho_1\{+\sqrt{(p^2+m^2)}\} = \rho_1 E \tag{2.2}$$

The expressions for the Poincaré group generators are

$$\mathbf{P}_C = \mathbf{P}; \qquad \mathbf{J}_C = \mathbf{J} \tag{2.3}$$

$$\mathbf{K}_C = -(1/2)(\mathbf{x}H_C + H_C\mathbf{x}) + t\mathbf{p} + H_C(\mathbf{S}\times\mathbf{p})/E(E+m) \tag{2.3a}$$

(ii) *E-Representation*

An operator to transform the C-representation into E-representation is obtained here. From (2.1), we have

$$\psi_C \to \psi = S\psi_C = [\textstyle\sum \delta_\mu^{(+)} B_\mu + \rho_1 \sum \delta_\mu^{(-)} C_\mu]\psi_C \tag{2.4}$$

$$\delta_\mu^{(\pm)} = \pm(m/4E)^{1/2}(mE+mp)^{-\mu}\,[(E+p)^{2\mu} \pm m^{2\mu}] \tag{2.4a}$$

In an extreme relativistic situation with which we are concerned here, the moving mass (m_v) of a particle with a velocity v will be infinitely greater than its rest mass m. Consequently, the ratio (m/m_v) and hence (m/p) would tend to zero. As a result, we get, for the extreme relativistic situation, $1 + m/p \to 1$. A similar consideration would show that for a non-relativistic situation, we will have $1 + p/m \to 1$. When this low momentum approximation is injected into the operator S, as given in (2.4), it becomes simply a unit operator, since $\delta_\mu^{(+)}(p/m \to 0) = 1$ and $\delta_\mu^{(-)}(p/m \to 0) = 0$. This means, while S transforms ψ_C into ψ, $S(p/m \to 0)$ transforms ψ_C into ψ_C itself. This suggests that the other case $S(m/p \to 0)$ would transform ψ_C into the extreme relativistic wave function which we denote by the symbol ψ_E. Accordingly we write $\psi_C \to \psi_E = R\psi_C = S(m/p \to 0)\psi_C$. Applying the high momentum approximation to the operator S, we get

$$R = \sum \delta_\mu (B_\mu - \rho_1 C_\mu) \tag{2.5}$$

where,

$$\delta_\mu = (m/4E)^{1/2} (2E/m)^\mu \tag{2.5a}$$

For the special case of spin-half, we get

$$R = (1/\sqrt{2}) [1 - \rho_1(\boldsymbol{\alpha}.\mathbf{p})/p] \tag{2.5b}$$

where

$$\boldsymbol{\lambda} = \boldsymbol{\alpha}/2 \tag{2.5c}$$

The operator given by (2.5b) is exactly the same as that used by Cini & Touschek (1958) in their treatment of the E-representation for the spin-half particles.

For every infinitesimal generator G_C acting on ψ_C there is a corresponding generator G_E acting on ψ_E being related to the former by a similarity transformation of the form $G_E = RG_C R^{-1}$ which enables us to obtain an expression for G_E once we know that for G_C. Using the expressions for R and R^{-1} we get‡

$$\mathbf{P}_E = \mathbf{P}; \qquad \mathbf{J}_E = \mathbf{J}; \qquad H_E = E \sum_{1/2}^{s} C_\mu \tag{2.6}$$

$$\mathbf{K}_E = t\mathbf{p} - \mathbf{x}H_E + \frac{i(4E^2 - m^2)}{4pE}\boldsymbol{\lambda} - \frac{i3m^2}{4p^3E}(\boldsymbol{\lambda}.\mathbf{p})\mathbf{p}$$
$$+ \frac{m^2 H_E(\mathbf{p} \times \mathbf{S})}{4p^2E^2} - \frac{\rho_1 m}{2p^2}(B_{1/2}\mathbf{S} + iC_{1/2}\mathbf{c}) \times \mathbf{p} \tag{2.6a}$$

where

$$\mathbf{c} = (\boldsymbol{\lambda} \times \mathbf{p})/p \tag{2.6b}$$

Equation (2.6) is the generalised E-representation Hamiltonian for arbitrary half integral spin, which in the special case of spin-half reduces to the expression $H_E = E(\boldsymbol{\alpha}.\mathbf{p})/p$ obtained by Cini & Touschek (1958). On the wave function ψ_E, H_E acts such that $i\partial\psi_E/\partial t = H_E\psi_E$ and this is the Schrödinger wave equation in the E-representation.

(iii) *Lorentz Invariant Scalar Product and Physical Assignments*

In the representation ψ_C the Lorentz invariant inner product is defined as $(\psi_C, \psi_C) = \int \psi_C^\dagger \psi_C d^3x$. The fact that the operators given by equations (2.3)

‡ To evaluate $R\mathbf{K}_C R^{-1}$ we make use of the expressions for the commutation relations like $[\mathbf{x}, B_\mu]$, $[\mathbf{x}, C_\mu]$; $[\boldsymbol{\lambda}, B_\mu]$ and $[\boldsymbol{\lambda}, C_\mu]$ obtained by M. Seetharaman, J. Jayaraman and P. M. Mathews (1971), *Journal of Mathematical Physics*, **12**, 835.

are Hermitian ($G_C^\dagger = G_C$) ensures the invariance of the above product under the transformations of the Poincaré group in the sense $(\psi_C, \psi_C) = (\psi_C', \psi_C')$ where ψ_C' is the transformed wave function. A similar Lorentz invariant scalar product can be obtained for E-representation by replacing ψ_C by $R^{-1}\psi_E$ as shown below.

$$(\psi_E, \psi_E) = \int (R^{-1}\psi_E)^\dagger R^{-1}\psi_E \, d^3x = \int \psi_E^\dagger M_E \psi_E \, d^3x \tag{2.7}$$

where the metric operator

$$M_E = (R^{-1})^\dagger R^{-1} = \sum (m/2E)^{2\mu-1} B_\mu \tag{2.7a}$$

With respect to the scalar product given in (2.7), the expectation value of an operator A in a state ψ_E is defined by

$$\langle A \rangle = (\psi_E, A\psi_E) = \int \psi_E^\dagger M_E A \psi_E \, d^3x \tag{2.8}$$

For the operator A to be an observable, the above scalar product must be real in the sense $(\psi_E, A\psi_E) = (A\psi_E, \psi_E)$. This is possible only when $A^\dagger M_E = M_E A$. The conventional operators $\mathbf{x}$ for the position and $\mathbf{S}$ for the spin do not meet this requirement and hence are not observables in E-representation. We therefore obtain here suitable E-representation operators $\mathbf{X}_E$ for the position and $\mathbf{S}_E$ for the spin.

It has been shown (Mathews, 1966a) that the operators $\mathbf{x}$ and $\mathbf{S}$, which have all the required properties to be observables in the C-representation, can be expressed in terms of the generators of the Poincaré group as,

$$\begin{aligned}\mathbf{x} = t\mathbf{p}H_C E^{-2} - \mathbf{K}_C H_C E^{-2} + i\mathbf{p}/2E^2 \\ + [m(E+m)]^{-1}[(\mathbf{J}\times\mathbf{p})H_C + (\mathbf{K}_C \times \mathbf{p})\times\mathbf{p}]H_C E^{-2}\end{aligned} \tag{2.9}$$

$$\mathbf{S} = (E/m)\mathbf{J} - [m(E+m)]^{-1}(\mathbf{J}.\mathbf{p})\mathbf{p} + (Em)^{-1}(\mathbf{K}_C \times \mathbf{p})H_C \tag{2.9a}$$

By making a similarity transformation with R we get,

$$\begin{aligned}\mathbf{X}_E = R\mathbf{x}R^{-1} = \mathbf{x} + i\mathbf{p}/2E^2 + \frac{(\mathbf{S}\times\mathbf{p})}{m(E+m)} - \frac{i(\boldsymbol{\lambda}.\mathbf{p})\mathbf{p}H_E}{pE^3} \\ + \frac{[m^2 H_E \mathbf{S} + i(4E^3 - Em^2)\mathbf{c}]\times\mathbf{p}H_E}{4p^2E^3 m} \\ + \frac{\rho_1(B_{1/2}\mathbf{S} + iC_{1/2}\mathbf{c})\times\mathbf{p}H_E}{2p^2E}\end{aligned} \tag{2.10}$$

$$\begin{aligned}\mathbf{S}_E = (E/m)\mathbf{S} - \frac{(\mathbf{S}.\mathbf{p})\mathbf{p}}{m(E+m)} + \frac{[i(4E^3 - Em^2)\boldsymbol{\lambda} - m^2 H_E \rho_3 \mathbf{c}]\times\mathbf{p}H_E}{4pE^3 m} \\ + \frac{\rho_1(iC_{1/2}\boldsymbol{\lambda} - B_{1/2}\rho_3\mathbf{c})\times\mathbf{p}H_E}{2pE}\end{aligned} \tag{2.10a}$$

For spin-half equations (2.10) and (2.10a) reduce to

$$\mathbf{X}_E = \mathbf{x} + \frac{i\rho_1 \boldsymbol{\alpha}}{2p} - \frac{i\rho_1(\boldsymbol{\alpha}.\mathbf{p})\mathbf{p}}{2p^3} - \frac{(\mathbf{S}\times\mathbf{p})}{p^2} \tag{2.11}$$

$$\mathbf{S}_E = \frac{(\mathbf{S}.\mathbf{p})\mathbf{p}}{p^2} - \frac{i\rho_1(\boldsymbol{\alpha}\times\mathbf{p})}{2p} \tag{2.11a}$$

which agree with those obtained by Mathews & Sankaranarayanan (1961, 1962, 1964) in their treatment of the E-representation.

3. *C and E-Representations for Integer Spins*

In deriving the results in C- and E-representations for integer spins, the only change we have to note is that the Hamiltonian in the D-representation is given not by (1.3), but by (1.3a). With this understanding, it is not difficult to see that in the C-representation the expressions for the Hamiltonian and the inner product are of the forms $H_C = \rho_3 E$ and $(\psi_C, \psi_C) = \int \psi_C^\dagger \rho_3 \psi_C d^3x$ and the operator representing space inversion is not ρ_1 but the unit matrix.‡ Moreover, the operator which takes the C-representation to the D-representation is found to be of the form§

$$\psi_C \to \psi = S\psi_C \tag{3.1}$$

$$S = \sum_0^s \delta_\mu^+(1+\rho_1)B_\mu + \rho_3 \sum_0^s \delta_\mu^-(1+\rho_1)C_\mu \tag{3.1a}$$

$$\delta_\mu^{(\pm)} = (m/8E)^{1/2}(mE+mp)^{-\mu}[(E+p)^{2\mu} \pm m^{2\mu}] \tag{3.1b}$$

$$S^{-1} = \sum_0^s \beta_\mu^+(1+\rho_1)B_\mu + \rho_3 \sum_0^s \beta_\mu^{(-)}(1+\rho_1)C_\mu \tag{3.1c}$$

$$\beta_\mu^{(\pm)} = (E/2m)^{1/2}(mE+mp)^{\mu}[(E+p)^{2\mu} \pm m^{2\mu}]^{-1} \tag{3.1d}$$

Applying the high momentum approximation $1 + m/p \to 1$ to the operators S and S^{-1} given in equations (3.1) we get

$$\psi_C \to \psi_E = S(m/p \to 0)\psi_C = R\psi_C \tag{3.2}$$

$$R = \sum \delta_\mu[(1+\rho_1)B_\mu + \rho_3(1+\rho_1)C_\mu];$$
$$\delta_\mu = (m/8E)^{1/2}(2E/m)^\mu \tag{3.2a}$$

$$R^{-1} = \sum \beta_\mu[(1+\rho_1)B_\mu + \rho_3(1+\rho_1)C_\mu];$$
$$\beta_\mu = (E/2m)^{1/2}(m/2E)^\mu \tag{3.2b}$$

‡ However, in the E-representation the space inversion is represented by ρ_1 only. Apart from this, it should also be mentioned that the operators representing the space inversion and charge conjugation anticommute for half integral spins and commute for integral spins.

§ The observation by P. M. Mathews *et al.*, *Nuovo Cimento* (1967), **50**, 339, that the Hamiltonian given in (34) of their paper and (1.3a) of this paper can be brought to a diagonal form $\rho_3 E$ through a similarity transformation provided the clue for the author to determine the operator S given in (3.1) and its inverse.

Finally, the relevant expressions for the Poincaré generators and Lorentz invariant scalar product in this case are found to be of the forms†

$$\mathbf{P}_E = \mathbf{P}; \qquad \mathbf{J}_E = \mathbf{J}; \qquad H_E = E \sum_{0}^{s} C_\mu \tag{3.3}$$

$$\mathbf{K}_E = t\mathbf{p} - \mathbf{x}H_E + \frac{i(4E^2 - m^2)\boldsymbol{\lambda}}{4pE} - \frac{i3m^2(\boldsymbol{\lambda}.\mathbf{p})\mathbf{p}}{4p^3 E} + \frac{m^2 H_E(\mathbf{p} \times \mathbf{S})}{4p^2 E^2} + \frac{im^2 \rho_1(B_0 \boldsymbol{\lambda} + i\rho_3 C_0 \mathbf{c})}{4pE} \tag{3.3a}$$

$$(\psi_E, \psi_E) = \int \psi_E^\dagger M_E \psi_E d^3 x; \qquad M_E = \sum (m/2E)^{2\mu-1} C_\mu \tag{3.3b}$$

With respect to the inner product given in (3.3b) the operator $\mathbf{X}_E$ for the position and $\mathbf{S}_E$ for the spin are found to be

$$\mathbf{X}_E = \mathbf{x} + i\mathbf{p}/2E^2 + \frac{(\mathbf{S} \times \mathbf{p})}{m(E+m)} - \frac{i(\boldsymbol{\lambda}.\mathbf{p})\mathbf{p}H_E}{pE^3} + \frac{[m^2 H_E \mathbf{S} + i(4E^3 - Em^2)\mathbf{c}] \times \mathbf{p}H_E}{4p^2 E^3 m} - i\rho_1 m(B_0 \boldsymbol{\lambda} + i\rho_3 C_0 \mathbf{c}) H_E/4pE^2 \tag{3.4}$$

$$\mathbf{S}_E = (E/m)\mathbf{S} - \frac{(\mathbf{S}.\mathbf{p})\mathbf{p}}{m(E+m)} + \frac{[i(4E^3 - Em^2)\boldsymbol{\lambda} - m^2 H_E \rho_3 \mathbf{c}] \times \mathbf{p}H_E}{4pE^3 m} + i\rho_1 m(B_0 \mathbf{c} - i\rho_3 C_0 \boldsymbol{\lambda}) H_E/4E^2 \tag{3.4a}$$

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† The Hamiltonian given in (3.3) has a 'drawback' in the sense that C_0 is not a well-defined quantity even though B_0 is well defined. This 'drawback', which is the case also with the D-representation Hamiltonian (1.3a) of this paper, has been explained by M. Seetharaman, J. Jayaraman and P. M. Mathews (1970), *Nuclear Physics*, **19B**, 625, as implying that for any integral spin, the first time derivative of the zero-helicity part of the wave function is indeterminate.

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